Mathematical Play: The Game of Cycles as a Math Circles Activity

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Outline

- What is the Game of Cycles?
- What We Did
- What Went Well
- What We're Revising
- What Next?

Rules of the Game of Cycles

Let's learn how to play the Game of Cycles[1].

- The board for the game consists of **vertices** (dots) and **edges** (lines) connecting those dots which don't intersect. We call the enclosed two-dimensional regions **cells.**

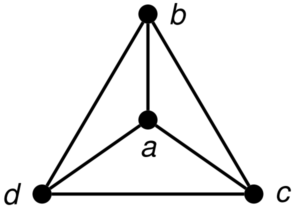

Figure: Example Board

[1] introduced by Su in Mathematics for Human Flourishing, 2020

Rules of the Game of Cycles

- Two players take turns marking an unmarked edge with an arrow pointing along it in one direction. For each vertex, we're not allowed to have all arrows pointing in (called a **sink**) or all pointing out (called a **source**).

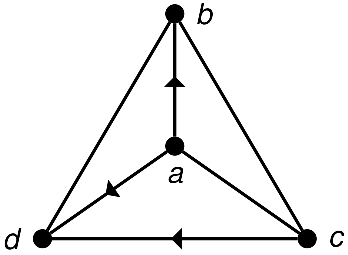

Figure: In-Progress Game

Rules of the Game of Cycles

- It does not matter who marks each edge.
- Players must make a move if there is one available.
- The game ends when a player creates a **cycle cell**, which is a cell where the edges bordering it are all marked with the same orientation (clockwise or counter-clockwise). The player to make a cycle cell is the winner.
- The game also ends if there are no legal moves remaining, and the last player to play is the winner.
- Have fun!

Let's Play!

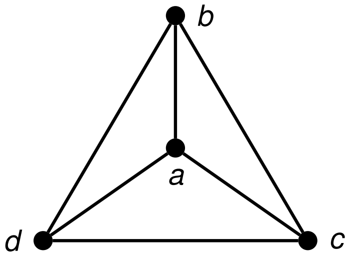

Figure: Example Board

Play at least 10 times, alternating who goes first. See if one of the players seems to have an advantage!

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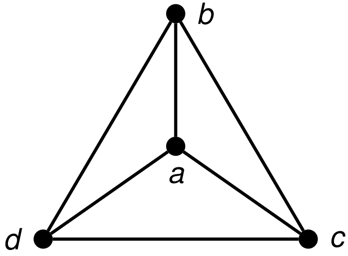


Figure: Example Board

Play at least 10 times, alternating who goes first. See if one of the players seems to have an advantage!

What We Did:

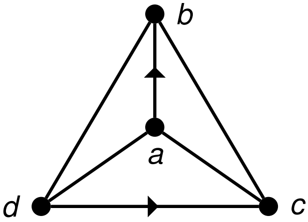
- We spent a little longer on that introduction in the session, but it still only took about 5 minutes. At that point, we were ready to play!
- We made worksheets with roughly a dozen copies of the first game board, and distributed them for the first game.
- This first board is nice because it is not immediately obvious which player has a winning strategy, nor what that strategy should be.
- We then gave attendees about 10 minutes to pair up and play the game, playing as both first and second player, seeing who won.
- Session leader wanders the room, helping attendees avoid making illegal moves (we'll come back to this...).

What We Did:

- After everyone had played a couple games as both first and second player, we got back together.
- Leader asked if anyone had predictions on if one of the players had an advantage, took a vote on it, and one representative from each side came to play at the board.
- At this point, leader gave a short (5 minute) introduction to the properties that make a combinatorial game and to Zermelo's Theorem, which states that one of the players **must** have a winning strategy.
- We then returned to the earlier board, and went to work.

What We Did:

- We talked a bit about symmetry, and how using the symmetries of the board it is always possible for Player 2 to make the board look as follows after their first move:



- From here, the limited options make the game tree easy to explicitly draw out, and the second player winning strategy can be proven. This was roughly another 10 minutes.

What We Did:

- At this point, there were about 20 minutes left in the session, and the rest of the activities were driven by the audience and how things had gone to this point.
- In all cases, more boards were introduced,along with the general concept of wheel graphs. We specifically focused on the two below:

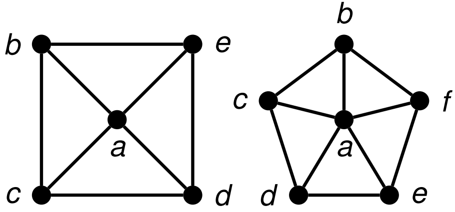


Figure: Wheel Graphs

What We Did:

- Attendees played on the 4-Spoked wheel graph and again tried to determine which player had a winning strategy.
- If time, attendees also played the game on the 5-Spoked wheel graph.
- We concluded by sharing that the winning strategy on the 5-Spoked wheel graph had been obtained in Summer 2023, and that there were many boards yet to be solved!

What Went Well:

- Attendees of all sessions understood the basics of how to play and when the game was over, very quickly. They had fun playing the game.
- They did a good job following along with the idea of the game tree, and how to always win on the first board when they were second player. A number of students talked about how they wanted to play the game with friends/family and always win.
- Generally were intrigued at the fact that this is an active area of research, and that there are unanswered questions in math.

What We're Revising:

- Next time we do this, we will spend a bit more time showing examples of illegal moves (that create sinks or sources), since a large number of them could be seen in the played games. Possibly some in-progress boards where we ask which move is allowed?
- While we want to make sure we get to the 5-spoke wheel, since that's where the recent research was done, we may skip the even-spoke wheel discussion entirely.
- We may spend more time on symmetry, and what boards are equivalent, since the ability to make any board look like the one displayed earlier is what makes the game tree only have 3 branches.
- We plan to include a board that's a first player win, to show second player is not always the winner.

What Next?

- If we do include the even-spoke wheel, how much strategy do we give away? Do we do it in the broader context (of symmetric graphs subject to certain caveats) or not?
- Is it worth including other classes of boards? You can play on any planar graph, and the current iteration of the session focuses entirely on wheel graphs.
- Include a survey for feedback afterwards.
- Other thoughts you have?

References

- Michael H. Albert, Richard J. Nowakowski, and David Wolfe. *Lessons in play: an introduction to combinatorial game theory*. AK Peters, 2007.
- Francis Su. *Mathematics for Human Flourishing*. Yale University Press, 2020.
- Alvarado, Averett, G-, Jackson, Karker, Marciniak, Su, Walker. *The Game of Cycles*. arXiv:2004.00776
- Campione, G-. *Playing the Game of Cycles on Wheel Graphs*. in preparation

Thank You

- Any questions?

Winning the 5-Spoked Wheel Graph

If we get the game into the position with the two edges described earlier marked:

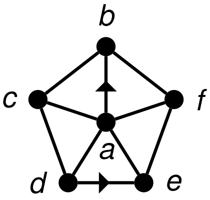


Figure: 5-Spoked Wheel Graph after 2 moves

There are only 10 legal Player 1 non-death moves.

And all have a response that Player 2 can make to get to a win!

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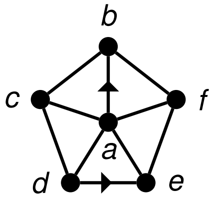

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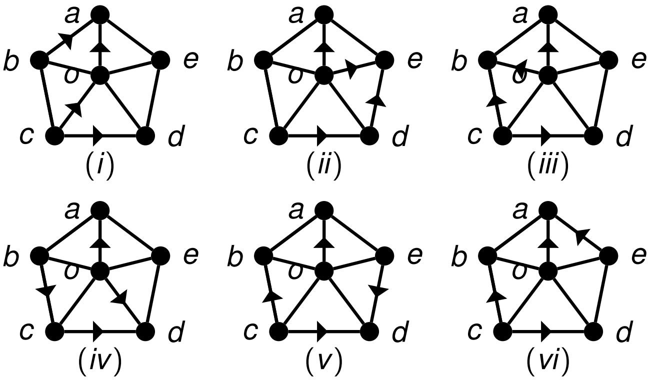

Figure: 4-move Boards that end in a P2 win

Winning the 7-Spoked Wheel Graph

We again start with the two edges described earlier:

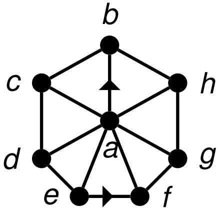

Figure: 7-Spoked Wheel Graph after 2 moves

From here there are 18 legal Player 1 non-death moves. And through careful consideration, Player 2 has a winning response to all of them!